A Derivation of Amortization — *Bret D. Whissel*

This is my derivation of the formula for amortization. The goal is to find a payment amount, x, which pays off the loan principal, P, after a specified number of payments, N. We start with some variable definitions:

P	The principal borrowed
N	The number of payments
i	The fractional (periodic) interest rate
P_j	The principal part of payment j
I_j	The interest part of payment j
B	A final balloon payment
x	The regular payment

Assuming that all payments (excluding an optional final balloon payment) are the same amount, a payment x consists of its interest part and its principal part:

$$x = I_j + P_j \tag{1}$$

$$\begin{array}{ll} I_1 = iP & P_1 = x - I_1 \\ I_2 = i(P - P_1) & P_2 = x - I_2 \\ I_3 = i(P - P_1 - P_2) & P_3 = x - I_3, \text{ etc.} \end{array}$$

This schedule states that the payment x includes interest on all of the remaining principal, including that which is part of the current payment. The first payment, therefore, includes an interest payment on the total borrowed, which defines the minimum payment. (If we are to make any progress toward paying off the loan, we must pay more than the amount iP.)

The P_j's may be rewritten into a recurrence relation:

$$P_1 = x - iP$$

$$\begin{aligned} P_2 &= x - i(P - P_1) \\ &= x - i\left[P - (x - iP)\right] \\ &= x - iP + ix - i^2 P \\ &= (x - iP)(1 + i) \end{aligned}$$

$$\begin{aligned} P_3 &= x - i(P - P_1 - P_2) \\ &= x - i\left[P - (x - iP) - (x - iP + ix - i^2 P)\right] \\ &= x + 2ix + i^2 x - iP - 2i^2 P - i^3 P \\ &= x(1 + i)^2 - iP(1 + i)^2 \\ &= (x - iP)(1 + i)^2 \end{aligned}$$

In general, we will find that

$$P_j = (x - iP)(1 + i)^{j-1}. \tag{2}$$

If there is to be a balloon payment, then the final payment will consist of the final principal payment P_f and interest on that principal iP_f so that $B = P_f + iP_f$. Rewriting P_f in terms of B gives $P_f = B/(1 + i)$.

Next, we define an equation which uses these ideas:

$$B + Nx = P + \sum_{j=1}^{N} I_j + i\left(\frac{B}{1+i}\right), \tag{3}$$

or in English, the sum of all the payments (left side) is equal to the principal borrowed plus all of the interest paid with regular payments plus interest paid on the balloon payment (right side). Note that if there will be no balloon payment ($B = 0$), then the B terms drop out.

Now we glue some more pieces together: replace I_j of equation (3) using the relationship given by eq. (1) and then substitute the recurrence identity of eq. (2):

$$B - \frac{iB}{1+i} + Nx = P + \sum_{j=1}^{N} \left[x - (x - iP)(1 + i)^{j-1}\right]$$

$$B - \frac{iB}{1+i} + Nx = P + Nx - (x - iP)\sum_{j=1}^{N}(1 + i)^{j-1}$$

$$P - B\left(1 - \frac{i}{1+i}\right) = (x - iP)\sum_{j=1}^{N}(1 + i)^{j-1}$$

Now we can see our way clear to solve for x:

$$x = \frac{P - B\left(1 - \frac{i}{1+i}\right)}{\sum_{j=1}^{N}(1 + i)^{j-1}} + iP. \tag{4}$$

The series form of eq. (4) can be rewritten without the series after a little transformation. First, we separate the summation and rewrite its limits:

$$x = \left[P - B\left(1 - \frac{i}{1+i}\right)\right]\frac{1}{\sum_{j=0}^{N-1}(1 + i)^{j}} + iP.$$

To simplify the transformation, we can substitute by letting $g = 1 + i$ so that the summation looks like $\sum_{j=0}^{N-1} g^j$. Next we multiply the series by $(1 - g)/(1 - g)$ so that all but the first and last terms drop out:

$$\frac{(1 - g)\sum_{j=0}^{N-1} g^j}{1 - g} = \frac{\sum_{j=0}^{N-1} g^j - \sum_{j=1}^{N} g^j}{1 - g} = \frac{1 - g^N}{1 - g}.$$

Since the series is originally in the denominator, we invert the transformed result, and then undo the substitution:

$$x = \left[P - B\left(1 - \frac{i}{1+i}\right)\right]\frac{1 - (1 + i)}{1 - (1 + i)^N} + iP. \tag{5}$$

Now we can expand and rearrange to taste:

$$x = i\left[\frac{P(1 + i)^N}{(1 + i)^N - 1} + \frac{B}{(1 + i) - (1 + i)^{N+1}}\right]. \tag{6}$$

Quod erat demonstrandum ("That which was to be shown"), otherwise known as Q.E.D. — BDW

Equations (5) and (6) solve for the payment amount, but either can be re-arranged to solve for any of the other variables, with the exception of i, the periodic interest rate. To date I have been unable to find an analytic solution for this variable, so the program invokes an iterative method to find successive approximations to the solution. □