Way back when I first figured out amortization for myself, I had solved the recurrence relationships for the principal component of payments. I had always been curious about the form of the recurrence relationships from the interest component of a payment.

As I was searching for a closed-form solution to an amortization-related problem, I had the opportunity to revisit this, and so I'm writing up the results of my doodles once again. I'm trying to be consistent with variable names over all these related documents, and so we have the following variables:

P	The principal borrowed
N	The total number of scheduled payments
i	The fractional (periodic) interest rate
P_j	The principal part of payment j
I_j	The interest part of payment j
x	The regular periodic payment

Here are some additional variables:

R_j	The principal remaining after j payments
A_j	The total interest paid out after j payments

In the original "Derivation of Amortization" document, I defined the relationship of interest and principal components of a payment x as follows:

$$x = I_j + P_j \qquad [1]$$

As defined in "Derivation", the interest component of a payment is calculated on all of the as yet unpaid loan principal. In the document "Some Additional Payment Tweaks", I derived an equation to calculate the principal remaining after a certain number of payments had been made. Here's a slightly modified version of that equation:

$$R_j = P - \frac{(x - iP)}{i}\left[(1+i)^j - 1\right]. \qquad [12*]$$

Note that $R_{j=0} = P$ is indeed defined correctly for our purposes, i.e., before we have made any payments, the entire principal balance remains.

Armed with this equation, we can re-express the payment components as follows (cf. eq. 1 of "Derivation of Amortization"):

$$\begin{array}{ll} I_1 = iR_0 & P_1 = x - I_1 \\ I_2 = iR_1 & P_2 = x - I_2 \\ I_3 = iR_2 & P_3 = x - I_3, \text{ etc.} \end{array}$$

With this formulation, it is easy to figure out what the interest component of any particular payment j will be.

If we want to find the total amount of interest that we have paid after j payments, we have this expression:

$$\begin{aligned} A_j = \sum_{k=1}^{j} I_k &= iR_0 + iR_1 + iR_2 + \cdots + iR_{j-1} \\ &= i\sum_{k=1}^{j} R_{k-1} \qquad (17) \end{aligned}$$

To come up with a new recurrence relation, we'll start by expanding eq. 17 to see what terms might be common and potentially re-grouped:

$$\begin{aligned} A_j = i\Bigg\{ & P - \frac{(x-iP)}{i}\left[(1+i)^0 - 1\right] \\ & + P - \frac{(x-iP)}{i}\left[(1+i)^1 - 1\right] + \cdots \\ & + P - \frac{(x-iP)}{i}\left[(1+i)^{j-1} - 1\right] \Bigg\} \end{aligned}$$

And now start to rearrange the terms:

$$\begin{aligned} A_j &= i\left[jP + j\frac{(x-iP)}{i} - \frac{(x-iP)}{i}\sum_{k=0}^{j-1}(1+i)^k \right] \\ &= ijP + j(x-iP) - (x-iP)\sum_{k=0}^{j-1}(1+i)^k \qquad (18) \end{aligned}$$

The series in eq. 18 is the same as in the "Payment Tweaks" document (see eq. 11), and so we may convert that to closed form directly:

$$A_j = ijP + j(x-iP) - (x-iP)\left[\frac{(1+i)^j - 1}{i}\right],$$

and then the final cleanup gives us

$$A_j = jx - \frac{(x-iP)}{i}\left[(1+i)^j - 1\right]. \qquad (19)$$

So we may now use eq. 19 if we need to calculate the interest which has been paid on a loan after j payments. (However, because we have rounded the payment x to the nearest whole penny, the quantity A_j may not exactly agree with a particular amortization schedule.)

If we analyze this equation, we can see that it makes sense: jx indicates the total amount paid so far (i.e., j payments of x each). From that quantity, we subtract the portion that's been put toward principal so far (again, refer to the derivation of eq. 12 in "Payment Tweaks"), with the result being the interest that has been paid. (In fact, I had come up with this formulation directly, but it's nice to see how it fell out as a natural consequence of the recurrence relation for interest.)

References include "A Derivation of Amortization" at `http://www.bretwhissel.net/amortization/amortize.pdf` and "Some Additional Payment Tweaks" at `http://www.bretwhissel.net/amortization/finextra.pdf`.